An Exterior Algebra Valued Tutte Function on Linear Matroids or their Pairs [3]

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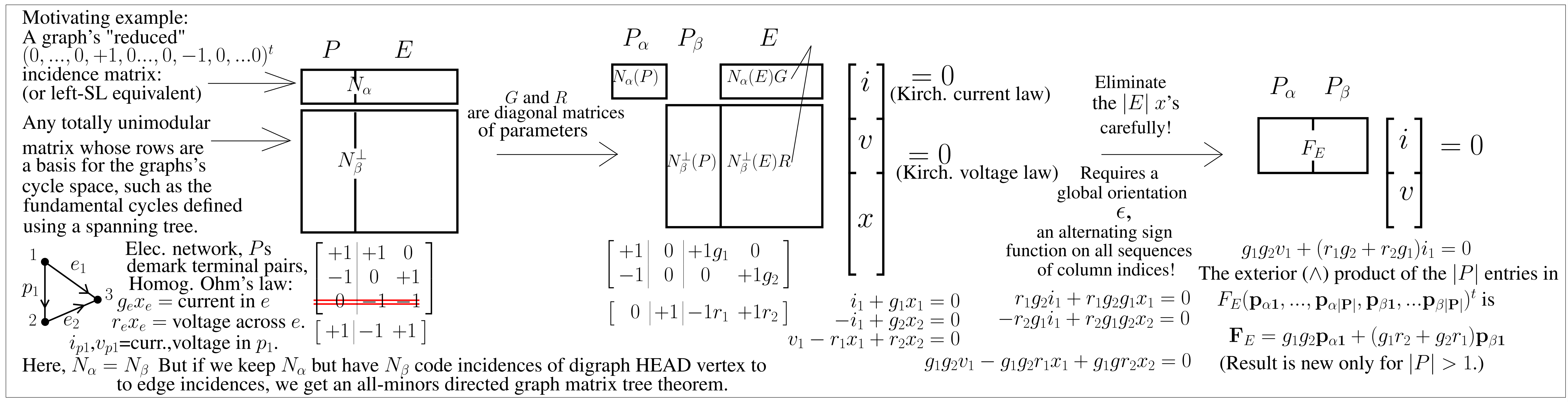
Overview



Motivating example:
A graph's "reduced" $(0, ..., 0, +1, 0..., 0, -1, 0, ...0)^t$ incidence matrix:
(or left-SL equivalent)

Any totally unimodular matrix whose rows are a basis for the graphs's cycle space, such as the fundamental cycles defined using a spanning tree.

Elec. network, Ps demark terminal pairs,
Homog. Ohm's law:
$g_e x_e$ = current in e
$r_e x_e$ = voltage across e.
i_{p1},v_{p1}=curr.,voltage in p_1.

$$\left[\begin{array}{c|cc} +1 & +1 & 0 \\ -1 & 0 & +1 \\ \cancel{0} & \cancel{-1} & \cancel{-1} \end{array}\right] \qquad \left[\begin{array}{c|cc} +1 & -1 & +1 \end{array}\right]$$

G and R are diagonal matrices of parameters

$$\left[\begin{array}{c|c|cc} +1 & 0 & +1g_1 & 0 \\ -1 & 0 & 0 & +1g_2 \end{array}\right] \qquad \left[\begin{array}{c|c|cc} 0 & +1 & -1r_1 & +1r_2 \end{array}\right]$$

$$\begin{bmatrix} i \\ v \\ x \end{bmatrix}$$

$= 0$ (Kirch. current law)

$= 0$ (Kirch. voltage law)

$$i_1 + g_1 x_1 = 0$$
$$-i_1 + g_2 x_2 = 0$$
$$v_1 - r_1 x_1 + r_2 x_2 = 0$$

Eliminate the $|E|$ x's carefully!

Requires a global orientation ϵ, an alternating sign function on all sequences of column indices!

$$r_1 g_2 i_1 + r_1 g_2 g_1 x_1 = 0$$
$$-r_2 g_1 i_1 + r_2 g_1 g_2 x_2 = 0$$
$$g_1 g_2 v_1 - g_1 g_2 r_1 x_1 + g_1 g r_2 x_2 = 0$$

$$\begin{bmatrix} i \\ v \end{bmatrix} = 0$$

$$g_1 g_2 v_1 + (r_1 g_2 + r_2 g_1) i_1 = 0$$

The exterior ($\wedge$) product of the $|P|$ entries in $F_E(\mathbf{p}_{\alpha\mathbf{1}}, ..., \mathbf{p}_{\alpha|\mathbf{P}|}, \mathbf{p}_{\beta\mathbf{1}}, ...\mathbf{p}_{\beta|\mathbf{P}|})^t$ is

$$\mathbf{F}_E = g_1 g_2 \mathbf{p}_{\alpha\mathbf{1}} + (g_1 r_2 + g_2 r_1)\mathbf{p}_{\beta\mathbf{1}}$$

(Result is new only for $|P| > 1$.)

Here, $N_\alpha = N_\beta$ But if we keep N_α but have N_β code incidences of digraph HEAD vertex to to edge incidences, we get an all-minors directed graph matrix tree theorem.

Theorem

Given N_α, N_β, of equal full row rank with columns labelled by $P \coprod E$, let $\mathbf{N}_\alpha$, $\mathbf{N}_\beta$ be the exterior products of their rows and dual $\mathbf{N}_\beta^\perp$ be constructed using (2).

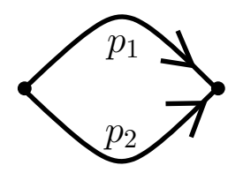
For all $e \in E$ such that $e \notin P$, with $E' = E \backslash e$,

$$\epsilon(PE)\mathbf{F}_E(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) = \epsilon(PE')\left(g_e \mathbf{F}_{E'}(\mathbf{N}_\alpha/e \;\; \mathbf{N}_\beta/e) + r_e \mathbf{F}_{E'}(\mathbf{N}_\alpha\backslash e \;\; \mathbf{N}_\beta\backslash e)\right)$$

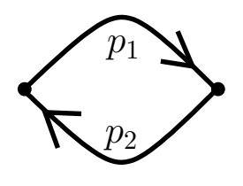
Similar identity for direct sums.

Summary

1. Usual parametrized Tutte functions F are valued in comm. rings.
2. Matrix Tree Theorem: The tree enumerator Tutte function is a determinant.
3. Relative (with respect to set P) [4] aka set P pointed [7], P-"ported" F [1]
$$F(N, P) = r_e F(N \setminus e) + g_e F(N/e)$$
only when non-loop non-coloop $e \notin P$.
4. Our $\mathbf{F}_E$ has values in an exterior algebra (w. anti-commutative multiplication.)
5. P will play the role of graph vertices, to generalize deleting any equal sized row and col. sets from the Laplacian in the All-Minors Matrix Tree Theorem.
6. Our Tutte function's values are exterior products of vectors. Thus they carry $\binom{2p}{p}$ determinants–each one is a Tutte function! Multiplication, occurring when N is a direct sum, is anti-commutative.
7. Unlike ordinary Tutte functions, these distinguish (at least trivial) orientations of the same matroid: F matrices with distinct $\mathbf{F}_E(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta)$, $E = \emptyset$, $|P| = 2$



$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (\mathbf{p}_{\alpha 1} + \mathbf{p}_{\alpha 2} \wedge (\mathbf{p}_{\beta 1} - \mathbf{p}_{\beta 2}) = -\mathbf{p}_{\alpha 1}\mathbf{p}_{\beta 2} + \mathbf{p}_{\alpha 2}\mathbf{p}_{\beta 1} + (\text{equal}) \cdots$$



$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (\mathbf{p}_{\alpha 1} - \mathbf{p}_{\alpha 2} \wedge (\mathbf{p}_{\beta 1} + \mathbf{p}_{\beta 2}) = \mathbf{p}_{\alpha 1}\mathbf{p}_{\beta 2} - \mathbf{p}_{\alpha 2}\mathbf{p}_{\beta 1} + (\text{equal}) \cdots$$

Catalog of Oriented Matroid operations on OM(N:Matrix) and on $\mathbf{N} = \wedge(\mathbf{rows}(N))$

Op is on: which are: case we use:	chirotopes $\chi : B \to \{0, \pm\}$ $\chi : B \mapsto \text{sign}(N[B])$	exterior products decomposibles in $\wedge$ $\mathbf{N} : B \mapsto \mathbf{N}[B]$
OPERATION		
deletion $\bullet\backslash A$	restriction	restriction
contraction $\bullet/A$	$\pm\chi' : B \mapsto \chi(BA)$	$\mathbf{N}/A : B \mapsto \mathbf{N}[BA]$ (1)
duality $\bullet^\perp$	$\pm\chi^\perp : B \mapsto \chi(\overline{B})\epsilon(\overline{B}B)$	$\mathbf{N}^\perp : B \mapsto \mathbf{N}[\overline{B}]\epsilon(\overline{B}B)$ (2)

$\mathbf{N}[B]$ is one component of a *distinguished representative vector* of the Plücker coordinates (which are projective!) for the row space of N.
We must choose some global orientation ϵ in order to define duality as an exterior alg. operation! ϵ is any alternating sign function on all finite sequences of elements.
The definitions of deletion, contraction and dual imply these commutations:
(S is a ground set for exterior product analogs of matroids, needed so the analog of matroids with loops will have the analog of coloops. $X \subseteq S$ and $S' = S\backslash X$.)

$$(\mathbf{N}\backslash X)^\perp = \epsilon(S')\epsilon(S'X)(\mathbf{N}^\perp/X) \tag{3}$$

$$(\mathbf{N}/X)^\perp = \epsilon(S')\epsilon(S'X)(-1)^{|X|r\mathbf{N}^\perp}(\mathbf{N}^\perp\backslash X) \tag{4}$$

Setup and proof outline

- See the statement of the Theorem. rank(N_α)+rank$(N_\beta^\perp) = |E|+|P|$. $P_\alpha, P_\beta \cong P$, $P_\alpha \cap P_\beta = \emptyset$.
- Weight (parameter) matrices $G = \text{diag}\{g_e\}_{e\in E}$, $R = \text{diag}\{r_e\}_{e\in E}$.
- Matrix with columns $P_\alpha \coprod P_\beta \coprod E$

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]$$

Define

$$F(L) = ((\binom{2p}{p})) - \text{tuple of determinants } L[Q_\alpha\overline{Q_\beta}E])$$

indexed by sequences $Q_\alpha\overline{Q_\beta} \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$, $\overline{Q_\beta} \subseteq P_\beta$, $|Q_\alpha\overline{Q_\beta}| = p = |P|$.
Translate into exterior algebra definitions:

$$\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix} := (\iota(\mathbf{N}_\alpha)(P_\alpha) + \iota_G(\mathbf{N}_\alpha(E))) \wedge (\upsilon(\mathbf{N}_\beta^\perp)(P_\beta) + \upsilon_R(\mathbf{N}_\beta^\perp)(E))$$
$$= (\iota_G(\mathbf{N}_\alpha) \wedge \upsilon_R(\mathbf{N}_\beta^\perp))$$

$$\mathbf{F}_E(\mathbf{L}) := \mathbf{L}/E = \sum_{Q_\alpha,\overline{Q_\beta}} \mathbf{L}[Q_\alpha\overline{Q_\beta}E]\mathbf{Q}_\alpha\overline{\mathbf{Q}_\beta}$$
$$= ((\iota(\mathbf{N}_\alpha)\backslash e(\text{no } \mathbf{e}) + g_e(\iota(\mathbf{N}_\alpha)/e) \wedge \mathbf{e})$$
$$\wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e(\text{no } \mathbf{e}) + r_e(\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}))/E$$

$$\text{2 of 4 terms} = \Big(r_e \quad \iota(\mathbf{N}_\alpha)\backslash e \wedge (\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}$$
$$\text{vanish} \quad + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha)/e) \wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e) \wedge \mathbf{e}\Big)/E$$

$$\mathbf{F}_E(\mathbf{L}) = \mathbf{L}/E = \Big(r_e \quad \iota(\mathbf{N}_\alpha\backslash e) \wedge (\upsilon(\mathbf{N}_\beta^\perp/e)) \wedge \mathbf{e}$$
$$+ g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha/e)) \wedge (\upsilon(\mathbf{N}_\beta^\perp\backslash e)) \wedge \mathbf{e}\Big)/E$$

$$= r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha\backslash e \\ \mathbf{N}_\beta^\perp/e \end{pmatrix} \wedge \mathbf{e}/E\right) + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha/e \\ \mathbf{N}_\beta^\perp\backslash e \end{pmatrix} \wedge \mathbf{e}/E\right)$$

$$(\mathbf{N}\backslash e)^\perp = \epsilon(S')\epsilon(S'e)(\mathbf{N}^\perp/e)$$

$$(\mathbf{N}/e)^\perp = \epsilon(S')\epsilon(S'e)(-1)^{|\{e\}|r\mathbf{N}^\perp}(\mathbf{N}^\perp\backslash e)$$

$$= \epsilon(S)\epsilon(S'e)\left[r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha\backslash e \\ (\mathbf{N}_\beta\backslash e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right) + g_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha/e \\ (\mathbf{N}_\beta/e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right)\right]$$

With $\mathbf{L}(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) = \mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix}$, and more sign calculations:

Definition. *For E, P sets written as ordered sequences,*

$$\mathbf{F}_E(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) = \mathbf{L}(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta)/E$$

History

Not well-known 1847 graph theory paper! Kirchhoff's [5] "Matrix Tree Theorem" paraphrased: The solution to the linear resistive electrical network (Kirchhoff's and Ohm's laws) problem is comprised of ratios of certain tree or forest enumerators.
Maxwell [6] expressed those problems with matrices. EEs call Kirchhoff's result "Maxwell's rule."
The theorem only for $\mathbf{N}_\alpha = \mathbf{N}_\beta$, with a lot of background and motivation appears in [3].
In [2] the setup is related to oriented matroids and their relation to non-linear electrical network problem well-posedness: Is $L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix}$ non-singular for all positive r_e, g_e?

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